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Informationally Complete Sets of Physical Quantities

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The notion of informational completeness is formulated within the convex state (or operational) approach to statistical physical theories and employed to introduce a type of statistical metrics. Further, a criterion for a set of physical quantities to be informationally complete is proven. Some applications of this result are given within the algebraic and Hilbert space formulations of quantum theory.

1. INTRODUCTION

In the past years the concept of quantum observable was generalized in a systematic and thorough way so as to cover what is called *unsharp observable* today. Quite surprisingly, the set of unsharp quantum observables shows features which so far were known only in the realm of classical physics: noncommuting observables can be made coexistent, that is, jointly measurable, by introducing a sufficient degree of unsharpness in a well-defined way. Moreover, there exist unsharp observables the measurement of which allows a unique determination of the state of the object. Such observables are called informationally complete, which property is the subject of the present paper.

The case of classical statistical theories is particularly simple insofar as all observables are coexistent, so that there exists a common joint observable which necessarily is informationally complete. Therefore, in the applications of our general results we shall be concerned mainly with the quantum case.

The quasiclassical features of unsharp and, in particular, informationally complete observables raise the hope that their consideration may contribute to a better understanding of the relation between quantum mechanics and macroscopic theories. A systematic common framework for classical and quantum statistical theories is provided by the convex state, or operational

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approach. Moreover, essential features of informationally complete observables can be described in very general terms within this approach. For these reasons we shall present our results within the convex state scheme.

In Section 2 a short summary of the basic structures of general statistical physical theories is given and the concept of informational completeness is formulated within this framework. Section 3 introduces a type of statistical metric induced by informationally complete observables. Further, such observables are shown to give rise to injective isometric embeddings of statistical theories into classical statistical structures. A characterization of informationally complete sets of physical quantities is given in Section 4, while Section 5 contains some applications of this result. Finally, in Section 6 the relevance of informationally complete observables in various fields of physics is briefly pointed out.

2. INFORMATIONALLY COMPLETE OBSERVABLES IN STATISTICAL STATE SPACES

Let $(\mathscr{V}, \mathscr{S})$ be a complete base norm space and $(\mathscr{B}, e)$ its complete dual order unit space (Alfsen, 1971). That is, $\mathscr{V}$ is a real Banach space with closed, generating, proper cone $\mathscr{V}^+$, the closed convex set $\mathscr{S}$ being a base of $\mathscr{V}^+$. The cone $\mathscr{V}^+$ makes $\mathscr{V}$ an ordered vector space by virtue of the partial ordering $\rho \leq \sigma$, defined as $\sigma - \rho \in \mathscr{V}^+$. The order unit e of $\mathscr{B} = \mathscr{V}^*$ is the positive linear functional on $\mathscr{V}$ satisfying $e(\rho) = \|\rho\| = 1$ for $\rho \in \mathscr{S}$. Here $\|\cdot\|$ is the base norm defined as

$$\|\rho\| = \inf\{\lambda \geq 0 : \rho \in \lambda \operatorname{conv}(\mathscr{S} \cup -S)\}$$

The base $\mathscr{S}$ is interpreted as the set of statistical states of a physical system; $\mathscr{V}$ is called the state space of the system. The partial ordering on $\mathscr{V}$ is naturally transferred into the dual space $\mathscr{B}$ via $x \leq y$ iff $x(\rho) \leq y(\rho)$ for all $\rho \in \mathscr{V}^+$. The order unit interval $\mathscr{E} = [0, e]$ of $\mathscr{B}$ is interpreted physically as the set of effects (Ludwig, 1983) representing the possible outcomes of the so-called simple experiments (i.e., experiments having just two elementary outcomes): in short, if $\rho \in \mathscr{S}$ and $x \in \mathscr{E}$, then the number $x(\rho) \in [0, 1]$ is the probability for the effect x in the state ρ. Making use of the natural embedding $\mathscr{V} \subset \mathscr{B}^*$, we shall also apply the notation $\rho(x)$ for $x(\rho)$.

By definition, a physical quantity (an observable) is a (normalized) effect-valued measure $A : \Sigma \to \mathscr{E}$ from a Boolean σ-algebra Σ into the set of effects. Usually Σ is taken to be a σ-algebra of subsets of a set Ω. In the spirit of a generalized spectral theory (Abbati and Mania, 1981) one may also introduce observables as certain elements in the closure (in a suitable topology) of the span of the range $R(A)$ of an effect-valued measure (Schroeck, 1988; Schroeck, 1989). In this sense we shall refer to the whole of $\mathscr{B}$ as the space of observables of a physical system and to the effect-valued measures as the effect observables.

An important task of physical experiments is the inference of the state of a physical system (or of an ensemble of systems) from the measurement statistics. This problem suggests the following definitions [which are systematically reviewed in Busch and Lahti (1989)]. Let $\mathscr{L}$, $\mathscr{L}' \subseteq \mathscr{B}$ be subsets of observables, $\mathscr{T} \subseteq \mathscr{S}$ a subset of states. Two states ρ, σ are called *equivalent* with respect to $\mathscr{L}$ iff $\rho(x) = \sigma(x)$ for all $x \in \mathscr{L}$. The corresponding equivalence classes are denoted as $[\rho]^{\mathscr{L}}$. Sets $\mathscr{L}$ and $\mathscr{L}'$ are called *informationally equivalent* (with respect to $\mathscr{T}$) iff $[\rho]^{\mathscr{L}} = [\rho]^{\mathscr{L}'}$ for all $\rho \in \mathscr{S}$ (resp. $\rho \in \mathscr{T}$) (Ali and Doebner, 1976). $\mathscr{L}$ is said to be *informationally complete* (with respect to $\mathscr{T}$) iff $[\rho]^{\mathscr{L}} = \{\rho\}$ for all $\rho \in \mathscr{S}$ (resp. $\rho \in \mathscr{T}$) (Ali and Prugovecki, 1977*a*; Prugovecki, 1977). Clearly, $\mathscr{L} = \mathscr{E}$ is informationally complete (as the span of $\mathscr{E}$ is $\mathscr{B}$). But in view of the limited experimental possibilities, it would be desirable to find smaller sets of effects which are informationally complete. In particular, an effect observable is called informationally complete if its range is informationally complete. Such observables are known in quantum mechanics (cf. Section 5); but it is easy to see that no spectral measure is informationally complete (Busch and Lahti, 1989).

We note that any effect observable A induces in a canonical way the structure of a reduced statistical state space with respect to which it is informationally complete. This reduced state space is constituted by the equivalence classes $[\rho]^A \doteq [\rho]^{R(A)}$ [$R(A)$ denoting the range of A, $\rho \in \mathscr{S}$], which form a convex set by virtue of the operation

$$\lambda[\rho]^A + (1-\lambda)[\sigma]^A \doteq [\lambda\rho + (1-\lambda)\sigma]^A$$

Typical examples of such a structure are quantum mechanical superselection rules, or sets of macroscopically distinguishable states in statistical physical theories. In both cases there are classical, resp. macroscopic, observables which induce the reduction of the state spaces.

In general, a set $\mathscr{L} \subseteq \mathscr{B}$ is informationally complete exactly if it is informationally equivalent to $\mathscr{E}$. We shall derive a relation between $\mathscr{L}$ and $\mathscr{B}$ as a necessary and sufficient condition for $\mathscr{L}$ to be informationally complete. Let $\mathscr{L} = R(A)$ be the range of an informationally complete effect observable A; then the *statistics* $\{\rho(x) \colon x \in R(A)\}$ uniquely determines the state $\rho \in \mathscr{S}$. But the knowledge of ρ allows one to calculate the statistics and expectation values of all observables. Thus, it is clear that the range $R(A)$ of an informationally complete effect observable entails the whole of $\mathscr{B}$ in some sense. It is the task of this paper to elucidate in which sense this is true. Furthermore, the one-to-one correspondence between the statistics and the states provides a method of constructing *classical representations* of general statistical state spaces.

To begin with, we investigate some topological implications of the notion of informational completeness giving rise to a class of statistical metrics.

3. STATISTICAL METRIC

In the following $\mathscr{L}$ denotes a subset of $\mathscr{B}$. We start with a trivial reformulation of the definition of informational completeness.

Lemma 1. The following are equivalent:
(i) $\mathscr{L}$ is informationally complete.
(ii) $\forall \rho \in \mathscr{V} \{\forall x \in \mathscr{L} \cup \{e\}\, \rho(x)=0 \Rightarrow \rho = O\}$.

Proof. Put $\rho=\rho_+ - \rho_-$ with $\rho_+, \rho_- \in \mathscr{V}^+$. *First, let* $\mathscr{L}$ be informationally complete. Asume $\rho(x)=0$ for all $x\in\mathscr{L}\cup\{e\}$, but $\rho\neq O$. Now $0=\rho(e)=\rho_+(e)-\rho_-(e)$, which implies that $\rho_+(e)=\rho_-(e)$. If $\rho(e)=0$, then $\rho_+(x)=0$ for all $x\in E$, and therefore $\rho_+=O$. As $\rho_+(e)=0$ is equivalent to $\rho_-(e)=0$, it follows that $\rho=O$, in contradiction to our assumption. Hence $\rho_+(e)\neq 0$, and thus both ρ_+ and ρ_- are nonzero. Now define $\hat{\rho}_+=\rho_+/\rho_+(e)$, $\hat{\rho}_-=\rho_-/\rho_-(e)\in\mathscr{S}$. It follows that $\hat{\rho}(x)\doteq\hat{\rho}_+(x)-\hat{\rho}_-(x)=\rho(x)/\rho_+(e)=0$ for all $x\in\mathscr{L}$ and $\hat{\rho}_+-\hat{\rho}_-=\rho/\hat{\rho}_+(e)\neq O$; thus $\hat{\rho}_+\neq\hat{\rho}_-$, in contradiction to the informational completeness of $\mathscr{L}$.

Conversely, let (ii) hold true, $\rho_1, \rho_2\in\mathscr{S}$. Assume $\rho_1(x)=\rho_2(x)$ for all $x\in\mathscr{L}\cup\{e\}$. Define $\rho=\rho_1-\rho_2$; then $\rho(x)=0$ for all $x\in\mathscr{L}\cup\{e\}$, hence $\rho=O$ and $\rho_1=\rho_2$. Thus, $\mathscr{L}$ is informationally complete. ■

Lemma 2. $\mathscr{L}$ is informationally complete iff $\mathscr{L}\cup\{e\}$ is informationally complete.

Proof. This is a direct consequence of the definition or of Lemma 1. ■

We define a semimetric on $\mathscr{S}$ via

$$d_{\mathscr{L}}:\quad \mathscr{S}\times\mathscr{S}\to\Re^+$$

$$(\rho,\sigma)\mapsto d_{\mathscr{L}}(\rho,\sigma)\doteq\sup\{|\rho(x)-\sigma(x)|/\|x\|: x\in\mathscr{L}\backslash\{0\}\}$$

Proposition 1. The following are equivalent:
(i) $\mathscr{L}$ is informationally complete.
(ii) $d_{\mathscr{L}}$ is a metric on $\mathscr{S}$.

Proof. The validity of the triangle inequality is straightforward. One easily checks that $d_{\mathscr{L}}(\rho,\sigma)=0$ is equivalent to $\rho(x)=\sigma(x)$ for all $x\in\mathscr{L}\backslash\{O\}$. Therefore, $d_{\mathscr{L}}(\rho,\sigma)=0$ implies that $\rho=\sigma$ for all $\rho,\sigma\in\mathscr{S}$ iff $\mathscr{L}$ is informationally complete. ■

This result shows that any informationally complete observable A induces a *statistical metric* $d_A \doteq d_{R(A)}$ on the set $\mathscr{S}$ of statistical states. Furthermore, if an observable B is finer than an observable A, that is, if $R(A) \subset R(B)$, then one has $d_A \le d_B$. In general it holds true that $d_{\mathscr{L}} \le d_{\mathscr{E}}$. These relations may be interpreted in the following way. The statistical metric associated to an informationally complete observable provides a means of distinguishing the states in $\mathscr{S}$. In this sense the state resolution is the better, the finer the considered observable is.

We define a seminorm on $\mathscr{V}$ via

$$\begin{aligned} \|\cdot\|_{\mathscr{L}}:\quad & \mathscr{V} \to \Re^+ \\ & \rho \mapsto \|\rho\|_{\mathscr{L}} \doteq \sup\{|\rho(x)|/\|x\| : x \in \mathscr{L} \setminus \{O\}\} \end{aligned}$$

Note that $d_{\mathscr{L}}(\rho, \sigma) = \|\rho - \sigma\|_{\mathscr{L}}$.

Proposition 2. The following are equivalent:
(i) $\mathscr{L}$ is informationally complete.
(ii) $\|\cdot\|_{\mathscr{L} \cup \{e\}}$ is a norm on $\mathscr{V}$.

Proof. This is a straightforward consequence of Lemma 1. ■

We note that the above statistical metric, resp. norm (Busch, 1987*a*), are generalizations of the *natural metric*, resp. *norm*, studied in Gudder (1973), Hadjisavvas (1981), and Jauch *et al.* (1968). While one has again $\|\cdot\|_{\mathscr{L}} \le \|\cdot\|_{\mathscr{E}}$, it is not known whether these norms are equivalent. In the case of the quantum mechanical state space the norm $\|\cdot\|_{\mathscr{E}}$ is equivalent to the base norm (i.e., the trace norm).

The statistical norm $\|\cdot\|_A \doteq \|\cdot\|_{R(A)}$ associated to an informationally complete observable A admits an interesting measure-theoretic interpretation. Let (Ω, Σ) be the value space of A; then due to the informational completeness of A any state ρ is represented in a unique way as a probability measure A_ρ on (Ω, Σ). In the context of quantum mechanics such *classical representations* of statistical states were introduced as phase space representations in Ali and Prugovecki (1977*a*); they will be analyzed in more general terms in a forthcoming work (Busch *et al.*, 1990). The point of interest in the present context is that the statistical (A-) norm of a state ρ coincides with the total variation norm of the associated measure, that is, $\|\rho\|_A = \|A_\rho\|$. Hence, the linear mapping $\rho \mapsto A_\rho$ is an isometry with respect to this norm.

In the following we formulate some preparatory statements for the main result of this paper.

Lemma 3. $\mathscr{L}$ is informationally complete iff span($\mathscr{L}$) is informationally complete.

Proof. One implication is obvious. To prove the reverse, assume that $\mathscr{L}$ is not informationally complete. Thus, by Lemma 1 there exists $\rho \in \mathscr{V}$, $\rho \neq O$, such that $\rho(x) = 0$ for all $x \in \mathscr{L} \cup \{e\}$. Then also $\rho(x) = 0$ for all $x \in \mathrm{span}(\mathscr{L})$, which shows that $\mathrm{span}(\mathscr{L})$ is not informationally complete either. ■

In the following $\mathscr{L}^{(\sigma)}$ denotes the $\sigma(\mathscr{B}, \mathscr{V})$-, or σ-weak closure of $\mathscr{L}$ in $\mathscr{B}$.

Lemma 4. $\mathscr{L}$ is informationally complete iff $\mathscr{L}^{(\sigma)}$ is informationally complete.

Proof. One implication is obvious. To show the reverse, assume that $\mathscr{L}$ is not informationally complete. There exists $\rho \in \mathscr{V}$ such that $\rho(x) = 0$ for all $x \in \mathscr{L} \cup \{e\}$ and $\rho \neq O$. Let $x \in \mathscr{L}^{(\sigma)}$. There is a net $(x_\lambda) \subset \mathscr{L}$ converging to x in the σ-weak topology. Thus, $\rho(x_\lambda - x) \to 0$. Since $\rho(x_\lambda) = 0$ for all λ, one obtains $\rho(x) = 0$. As this holds true for arbitrary $a \in \mathscr{L}^{(\sigma)}$, this set is not informationally complete either. ■

Lemma 5. $\mathscr{L}$ is informationally complete iff $(\mathrm{span}(\mathscr{L}))^{(\sigma)}$ is informationally complete.

Proof. This follows directly from Lemmas 3 and 4. ■

4. CRITERION FOR INFORMATIONAL COMPLETENESS

The following theorem provides a basis for a variety of applications, yielding either new results or new insight into known facts. Some examples will be given in Section 5.

Theorem 1. A subset $\mathscr{L}$ of the dual order unit space $\mathscr{B}$ of a complete base norm space $(\mathscr{V}, \mathscr{S})$ is informationally complete iff $\mathrm{span}(\mathscr{L} \cup \{e\})$ is σ-weakly dense in $\mathscr{B}$.

Proof. By Lemmas 2 and 5, the informational completeness of $\mathscr{L}$ is equivalent to that of $\mathscr{L}_e = [\mathrm{span}(\mathscr{L} \cup \{e\})]^{(\sigma)}$. Further, this set is informationally complete if and only if its annihilator $\mathscr{L}_e^{\perp} = \{O\}$. By well-known duality theorems (Rudin, 1973, Theorems 4.7 and 4.9) this last equality is equivalent to $\mathscr{L}_e = \mathscr{B}$. ■

We note that if $e \in [\mathrm{span}(\mathscr{L})]^{(\sigma)}$, then $\mathscr{L}$ is informationally complete iff $[\mathrm{span}(\mathscr{L})]^{(\sigma)} = \mathscr{B}$. Further, if $\mathscr{V}$ is a base norm space of finite dimension n, then a set $\mathscr{L} \subset \mathscr{B} = \mathscr{V}^*$ is informationally complete iff its span has dimension n [in case $e \in \mathrm{span}(\mathscr{L})$], resp., $n-1$ [in case $e \notin \mathrm{span}(\mathscr{L})$].

5. SOME APPLICATIONS

A realization of the base norm space and dual order unit space scheme suitable for physical applications is provided by the W^*-algebraic approach. Let $\mathscr{A}$ be a unital W^*-algebra and $\mathscr{A}_*$ its predual, the space of normal states on $\mathscr{A}$. Then one identifies $\mathscr{V} \doteq (\mathscr{A}_*)_h$, the Hermitian part of $\mathscr{A}$, $\mathscr{S} \doteq (\mathscr{A}_*)_1^+$, the normal states on $\mathscr{A}$, $\mathscr{B} \doteq \mathscr{A}_h$, and $\mathscr{E} \doteq \{x \in \mathscr{A}^+ : \|x\| \leq 1\}$. The order unit element e is given by the unit element I of $\mathscr{A}$.

We shall mostly be concerned with the quantum mechanical realization of a concrete W^*-system, given by $\mathscr{A} \doteq \mathscr{B}(\mathscr{H})$, the von Neumann algebra of bounded operators acting on a complex separable Hilbert space $\mathscr{H}$, in which case the predual $\mathscr{A}_*$ can be identified with $\mathscr{B}_1(\mathscr{H})$, the trace class. The order unit of $\mathscr{B}$ is given by the trace functional via $e(\rho) \doteq \mathrm{Tr}(\rho) = \rho(I)$, $\rho \in \mathscr{A}_*$. In general, the duality between $\mathscr{B}_1(\mathscr{H})$ and $\mathscr{B}(\mathscr{H})$ is represented as $x(\rho) = \mathrm{Tr}[x\rho]$ (for $x \in \mathscr{B}(\mathscr{H})$, $\rho \in \mathscr{B}_1(\mathscr{H})$). Finally, note that the σ-weak topology on $\mathscr{B}(\mathscr{H})$ coincides with the ultraweak operator topology.

Example 1. Let s_x, s_y, s_z be the spin operators acting as generators of an irreducible representation of the rotation group on the Hilbert space $\mathscr{H}_s = \mathscr{C}^n$, $n = 2s+1$, $s = 0, 1/2, 1, 3/2, \ldots$. Let $\mathscr{L}$ be the union of the ranges $R(E^{s_i})$ of the spectral measures of the s_i. As a consequence of Theorem 1, $\mathscr{L}$ is informationally complete iff $s < 1$ [note the error in Busch and Lahti (1989)].

Example 2. The Schroeck problem (Schroeck, 1981). We define a phase space observable as an effect-valued (or positive-operator-valued) measure on the Borel algebra $\mathscr{B}(\Gamma)$ of phase space $\Gamma \doteq \Re^2$ (endowed with the usual symplectic form γ) into $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ hosts an irreducible representation $U: x \mapsto U_x$ of the Weyl relations $U_x U_y = \exp[i\gamma(x, y)/2\hbar] U_{x+y}$. With $x = (q, p)$, $y = (q', p')$ we have $\gamma(x, y) = q \cdot p' - p \cdot q'$. The self-adjoint generators Q, P of U, defined through $U_{(q,p)} = \exp[i(q \cdot P - p \cdot Q)/\hbar]$, satisfy the canonical commutation relations. Let $d\mu(q, p) = dq\, dp/2\pi\hbar$ be the Haar measure on Γ (viewed as the locally compact phase space translation group). Finally, let $T_0 \in \mathscr{B}_1(\mathscr{H})$ be a positive operator of trace one. Then the following defines a covariant phase space observable (Prugovecki, 1986):

$$a: \quad \mathscr{B}(\Gamma) \to \mathscr{E}$$

$$Z \mapsto a(Z) = \int_Z T_{qp}\, d\mu(q, p)$$

$$T_{qp} = U_{(q,p)} T_0 U_{(q,p)}^{-1}$$

Such phase space observables represent simultaneous position–momentum

(Q, P) measurements obeying the uncertainty relation; they lead to irreducible phase space representations of quantum mechanics in terms of irreducible subspaces of $L^2(\Gamma, \mu)$ (Ali, 1985; Prugovecki, 1986). The informational completeness of a is equivalent to the condition $\mathrm{Tr}[T_{qp} \cdot T_0] \neq 0$ for all $(q, p) \in \Gamma$ (Prugovecki, 1986).

We note that any operator of the form

$$a(f) = \int_\Gamma f(q, p) T_{qp} \, d\mu(q, p) \in \mathscr{B}(\mathscr{H}), \qquad f \in L^\infty_{\mathfrak{R}}(\Gamma, \mu)$$

can be approximated (strongly) by means of elements from the span of the range $R(a) = a(\mathscr{B}(\Gamma))$ of a (Schroeck, 1988, 1989, 1990). Now the question is how large the set of such operators

$$\mathscr{A}(a) \doteq \{a(f) : f \in L^\infty_{\mathfrak{R}}(\Gamma, \mu)\}$$

is in $\mathscr{B}(\mathscr{H})$. Theorem 1 gives a partial answer:

$$(\mathscr{A}(a))^{(\sigma)} = \mathscr{B}(\mathscr{H}) \Leftrightarrow a \text{ is informationally complete}$$

The proof follows from the observation that

$$\mathrm{span}(R(a)) \subseteq \mathscr{A}(a) \subseteq [\mathrm{span}(R(a))]^{(\sigma)}$$

Finally, we note some further implications for the quantum theory of measurement, a more detailed discussion of which can be found in Busch and Lahti (1989).

Proposition 3. If $a : \Sigma \to \mathscr{B}(\mathscr{H})^+$ is an informationally complete effect observable, then it is totally noncommutative, i.e.,

$$\mathrm{com}(a) \doteq \{\varphi \in \mathscr{H} : a(X)a(Y)\varphi = a(Y)a(X)\varphi \text{ for all } X, Y \in \Sigma\} = \{0\}.$$

Proof. To verify this statement, we note that the following chain of equations holds:

$$\mathrm{com}(\mathscr{L}) = \mathrm{com}(\mathscr{L} \cup \{I\}) = \mathrm{com}(\mathrm{span}(\mathscr{L} \cup \{I\}))$$
$$= \mathrm{com}([\mathrm{span}(\mathscr{L} \cup \{I\})]^{(\sigma)})$$

Then applying Theorem 1 and observing that $\mathrm{com}(\mathscr{B}(\mathscr{H})) = \{0\}$ yields the desired result. In the above chain, only the last equality needs some (straightforward) limit considerations (if $\mathscr{H}$ is infinite-dimensional). ■

The next proposition presupposes some operational notions, explicit definitions of which can be found in Busch and Lahti (1989). In short, an instrument $\mathscr{I}$ is an operation-valued measure on some σ-algebra Σ, an operation is a positive linear mapping on the trace class describing the state change of a physical system under measurement. Every instrument $\mathscr{I}$ induces a unique effect observable a in a canonical way via $\mathrm{Tr}[\mathscr{I}(X)\rho]=\mathrm{Tr}[\rho a(X)]$ (for all $X\in\Sigma$, all $\rho\in\mathscr{S}$). Conversely, an observable a generally admits more than one instrument according to this rule. An instrument $\mathscr{I}$ is called repeatable if it satisfies $\mathrm{Tr}[\mathscr{I}(X)\mathscr{I}(Y)\rho]=\mathrm{Tr}[\mathscr{I}(X\cap Y)\rho]$ for all X, $Y\in\Sigma$, all $\rho\in\mathscr{S}$ (Davies, 1976).

Proposition 4. If a is an informationally complete effect observable, then it does not permit any repeatable completely positive instruments.

Proof. See Busch and Lahti (1989) for details. In short, by a theorem of Ozawa (1984), only discrete observables a admit completely positive repeatable instruments. Then the statement is a consequence of the preceding proposition and the fact that for discrete observables, repeatability implies $\mathrm{com}(a)$ to be at least two-dimensional. ■

These measurement-theoretic results illustrate the limitations on the measurability of informationally complete observables in quantum mechanics. These limitations are the price to be paid for the possibility of approaching a classical measurement situation, namely the simultaneous determination of all noncommuting observables, which necessarily can be achieved only in an approximate way (Busch and Lahti, 1989).

As a last application of Theorem 1 we generalize the above-mentioned statement concerning the informational incompleteness of spectral measures. An elementary proof using Hilbert space techniques consists in the following. Let A be a self-adjoint operator, E^A its spectral measure. Take any vector $\varphi\in\mathscr{H}$ which is not an eigenvector of A and define $\psi=\exp(iA)\varphi$. Then the states ρ, σ corresponding to the one-dimensional projection operators onto the subspaces spanned by φ and ψ are different but satisfy

$$\mathrm{Tr}[\rho a]=\mathrm{Tr}[\sigma a] \quad \text{for all} \quad a\in R(E^A) \quad (\text{and } \rho\neq\sigma)$$

Thus A, resp. E^A, is not informationally complete.

In the abstract algebraic setting we obtain the following more general result.

Proposition 5. Let a be a projection-valued measure from a Boolean σ-algebra into a W^*-algebra $\mathscr{A}$. If $\mathscr{A}$ is commutative, then a is informationally complete iff the $*$-subalgebra $\mathscr{A}(a)$ generated by $R(a)$ equals $\mathscr{A}$. If $\mathscr{A}$ is noncommutative, then a is not informationally complete.

Here the algebra $\mathscr{A}(a)$ generated by $R(a)$ is the σ-weak closure of the complex span of $R(a)$.

Proof. The statement follows immediately from Theorem 1 and the following facts, the proofs of which are quite straightforward: (1) $\mathscr{A}(a)=\mathscr{A}_h(a)+i\mathscr{A}_h(a)$, where $\mathscr{A}_h(a)$ is the σ-weak closure [with respect to

$(\mathscr{A}_h)_* = (\mathscr{A}_*)_h]$ of the real span of $R(a)$; (2) $\mathscr{A}(a)$ is an Abelian W^*-subalgebra of $\mathscr{A}$. ■

6. CONCLUDING REMARKS

To summarize, some mathematical aspects and physical applications of informationally complete observables are outlined in this paper. In particular, metrical and topological characterizations of informational completeness are given which yield statistical distances and the possibility of classical (phase space) representations of quantum mechanics.

The field of generalized quantum mechanics, particularly the theory of unsharp observables, is rapidly developing toward practical applications; for a short overview the reader is referred to Busch *et al.* (1989). Effect observables, and among them informationally complete phase space observables, are of great importance in quantum optical experimentation (Grabowski, 1990, de Muynck and Martens, 1990). Moreover, the relevance of informational completeness to signal analysis and processing has been demonstrated recently (Healy and Schroeck, 1988). Also, there exist proposals for realizing concrete experiments for measuring informationally complete polarization observables (Busch, 1987*b*; Busch and Schroeck, 1989).

The idea of informational completeness also finds natural applications in statistical physics. First, as mentioned in Section 2, the reduction of state spaces by means of superselection rules or macroscopic observables may be interpreted in terms of informational completeness. Further, the phase space representations of quantum mechanics provide a basis for a unified treatment of quantum and classical statistical mechanics (Ali and Prugovecki, 1977*b*). More generally and finally, the classical embedding technique sketched in Section 3 may shed new light on the relations between quantum mechanical many-particle theories and macroscopic theories (Ludwig, 1987). This topic should be the subject of future investigations based on the work of Busch *et al.* (1990) and Ludwig (1987).

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